THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 50, No. 6, July-August, 1971
Printed in U.S.A.

Crosstalk Caused by Scattering in Slab Waveguides

By D. MARCUSE

(Manuscript received March 1, 1971)

Light scattered in one optical fiber can be scattered back into a guided mode in a neighboring optical fiber. This type of scattering crosstalk is investigated in this paper for slab waveguides. However, the results for the slab waveguide case are upper limits on the crosstalk between round optical fibers. Scattering crosstalk is expressed in terms of guided mode radiation loss whose cause is the same scattering mechanism. It is thus not necessary to know the details of the scattering mechanism as long as the scattering loss, which is a measurable quantity, is known. In addition to the total radiation loss the crosstalk depends on the width of the radiation lobe in which the scattered energy escapes from the waveguide. The crosstalk is inversely proportional to the width of the radiation lobe and thus is larger when the radiation lobe is narrow. Scattering crosstalk can become serious if both guides have a systematic sinusoidal imperfection of the same mechanical frequency resulting in a radiation lobe of nearly zero width. In the absence of a systematic sinusoidal imperfection it can be concluded that scattering crosstalk between dielectric waveguides (optical fibers) is negligible if the radiation losses are tolerable. In this case it appears unnecessary to suppress crosstalk by means of a lossy medium between the waveguides.

I. INTRODUCTION

In an earlier paper[1] I discussed the crosstalk between optical dielectric waveguides based on the directional coupler mechanism. In that case, crosstalk was caused by the fact that the exponentially decaying field of one guide reaches the region of a neighboring guide. This type of crosstalk exists even if both guides can be considered to be perfect dielectric cylinders.

The present paper is devoted to a crosstalk mechanism of a different

type. A dielectric waveguide or optical fiber loses power by radiation if the guiding structure is imperfect in any way. This imperfection may consist of deviations in the perfect cylindrical geometry of the core-cladding interface or it may consist in random variations of the refractive index distribution. Some of the radiation, causing loss to the guided mode of one fiber, may reach a neighboring fiber and can there be scattered back into one of its guided modes. This type of crosstalk is thus caused by waveguide imperfections. However, just as in the case of the directional coupler effect, it is possible to link predictions of the effectiveness of this type of crosstalk mechanism to the loss it causes to the guided mode. This makes it possible to give very simple rules that link crosstalk to radiation loss (see equation (50)). It turns out that scattering crosstalk is no serious problem provided that the radiation loss of the guided mode (that is caused by the same mechanism) remains within acceptable limits.

Our treatment of the scattering crosstalk problem is simplified by limiting it to the case of the slab waveguide. It is intuitively clear that the crosstalk between slab waveguides must be stronger than crosstalk between round optical fibers. In fact, the crosstalk between parallel slab waveguides is independent of their separation while the crosstalk between round fibers must vary inversely proportional to their distance. Furthermore, our treatment of crosstalk starts out by assuming a definite scattering mechanism. That is, we assume that the core of either guide varies in width. However, this special assumption allows us to describe the more general case. No matter what the mechanism, scattering of light can be attributed to the individual Fourier components of either the core width variations or of the departure of any other quantity from its perfect value. Regardless of the particular mechanism each Fourier component causes a narrow radiation lobe to escape in a definite direction. The same mechanism that produced the radiation lobe in one guide is responsible for the capture of some of this radiation in the neighboring guide. It is thus immaterial by what mechanism the radiation and reconversion was produced. In particular, if we express the strength of the scattering mechanism by the radiation loss that it causes to the guided mode, we have a description of the crosstalk effect that is independent of the actual scattering mechanism. Realization of these few basic facts greatly simplifies the treatment of the scattering crosstalk problem.

The deterministic scattering theory can easily be extended to include statistical assumptions about the scattering mechanism. We are thus able to express the scattering crosstalk in terms of three parameters: the scattering mode loss, the ratio of waveguide length to free space

wavelength of the guided radiation, and the width of the radiation lobe.

II. CROSSTALK CAUSED BY SINUSOIDAL CORE THICKNESS VARIATION

As discussed in the introduction, we are basing the treatment of scattering crosstalk on the mechanism of core width variations. To begin with, we concentrate on crosstalk caused by sinusoidal variations of the core of both slab waveguides. The more general case of arbitrary core width variations can be treated by decomposing the arbitrary functions into Fourier series and utilizing the result for a single sine wave component. Finally, we express the crosstalk in terms of radiation losses suffered by each guided mode and are thus able to free ourselves of the particular scattering mechanism.

It is well known that dielectric waveguides possess two types of modes. At a given operating frequency there are a finite number of guided modes and a continuum of radiation modes. The coupling between these two types of modes caused by variations of the core thickness has been explored in earlier papers.[2,3] We can thus simply draw on these earlier results for our present purposes.

The even and odd radiation modes of the dielectric slab waveguide [equations (24) and (25) of Ref. 3] can be expressed in the following simple form outside of the core, $x_1 > d$ (Fig. 1):

$$\mathcal{E}_y^{(e)}(\rho) = \left(\frac{2\omega\mu P}{\pi\beta}\right)^{\frac{1}{2}} \cos\left[\rho(|x_1| - d) + \phi\right] e^{-i\beta z}, \tag{1}$$

$$\mathcal{E}_y^{(o)}(\rho) = \frac{x_1}{|x_1|}\left(\frac{2\omega\mu P}{\pi\beta}\right)^{\frac{1}{2}} \cos\left[\rho(|x_1| - d) + \Psi\right] e^{-i\beta z}. \tag{2}$$

As usual, the factor $\exp(+i\omega t)$ has been suppressed. The phases appearing in (1) and (2) are given by

$$\tan\phi = \frac{\sigma}{\rho}\tan\sigma d \tag{3}$$

and

$$\tan\psi = -\frac{\sigma}{\rho}\cot\sigma d, \tag{4}$$

with ($k = \omega\sqrt{\epsilon_o\mu_o}$)

$$\rho = (n_2^2k^2 - \beta^2)^{\frac{1}{2}} \tag{5}$$

and

$$\sigma = (n_1^2k^2 - \beta^2)^{\frac{1}{2}}. \tag{6}$$

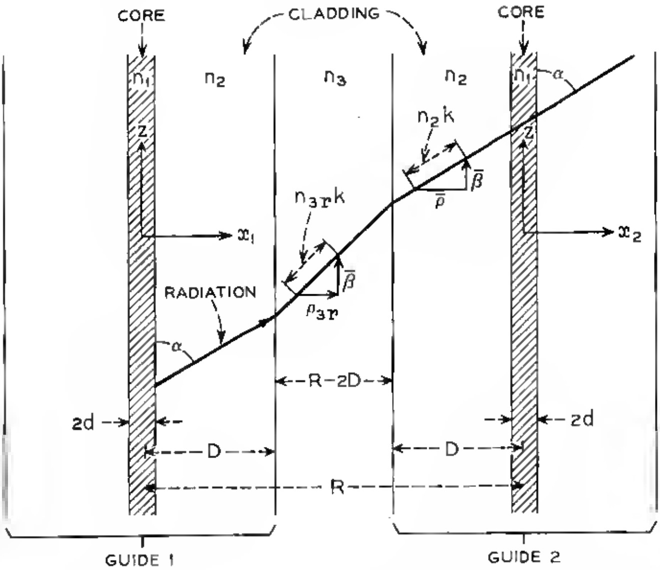

Fig. 1—Geometry of the two slab waveguides. The figure shows the definition of several parameters and indicates a scattered light beam (as a ray) going from guide 1 to guide 2.

The slab half-width is d; n_1 and n_2 are the refractive indices of core and cladding. The power carried by the radiation mode is defined by the orthogonality condition of the radiation modes:

$$\frac{\beta}{2\omega\mu}\int_{-\infty}^{\infty} \mathcal{E}_y^{(e)}(\rho)\,\mathcal{E}_y^{(o)}(\rho')\,dx = P\delta_{eo}\,\delta(\rho-\rho'). \tag{7}$$

δ_{eo} is the Kronecker delta symbol while $\delta(\rho - \rho')$ is Dirac's delta function. An even guided TE mode is coupled to only even radiation modes by a sinusoidal variation of the core thickness of the form

$$d_1 = d + a_1 \sin\theta_1 z. \tag{8}$$

According to Ref. 3 the radiation field excited by the sinusoidal thickness variation of the slab core is given by

$$E_y = -\frac{a_1k^2(2\omega\mu_o P)^{\frac{1}{2}}(n_1^2-n_2^2)\cos\kappa_1 d}{\pi\left(\beta_1 d + \dfrac{\beta_1}{\gamma_1}\right)^{\frac{1}{2}}}$$

$$\cdot\int_{\theta_1-\beta_1-n_2k}^{\theta_1-\beta_1+n_2k}\frac{\cos\sigma d\,\cos\,[\rho(x_1-d)+\phi]}{\{\rho^2\cos^2\sigma d+\sigma^2\sin^2\sigma d\}^{\frac{1}{2}}}\,e^{-i(\beta-\Delta/2)z}\,\frac{\sin\Delta\dfrac{z}{2}}{\Delta}\,d\Delta. \tag{9}$$

This formula holds for $d < x_1 < \infty$. The variable Δ is given by

$$\Delta = \theta_1 - (\beta_1 - \beta). \tag{10}$$

The parameter β_1 is the propagation constant of the even guided TE mode. The other parameters appearing in (9) are

$$\kappa_1 = (n_1^2k^2 - \beta_1^2)^{\frac{1}{2}} \tag{11}$$

and

$$\gamma_1 = (\beta_1^2 - n_2^2k^2)^{\frac{1}{2}}. \tag{12}$$

There are two differences between (9) and equation (27) of Ref. 3. We are now dealing with core thickness variations instead of a variation of only one of the faces of the slab core. The guide length L had to be replaced with the length coordinate z since we assume that the guide is very long and that we are looking at a field close to the fiber core and not at $z \gg L$ infinitely far from the sinusoidally perturbed section of length L. Finally, we have replaced the differential $d\rho$ of equation (27), Ref. 3 by

$$d\rho = -\frac{\beta}{\rho}\, d\Delta. \tag{13}$$

For large values of z the factor $(1/\pi)(\sin \Delta z/2)/\Delta$ approaches a delta function. We are thus permitted to take slowly varying functions out of the integral and are left with

$$\lim_{z\to\infty} \int_{-\infty}^{\infty} \cos\,[\rho(x_1 - d) + \phi]e^{-i(\beta-\Delta/2)z} \frac{\sin \Delta \dfrac{z}{2}}{\Delta}\, d\Delta$$
$$= \frac{\pi}{2} \exp\,\{-i[\bar{\beta}z + \bar{\rho}(x_1 - d) + \phi]\}. \tag{14}$$

The limits on the integral were extended from $-\infty$ to $+\infty$. This does not change its value since only the immediate vicinity of $\Delta = 0$ gives a contribution. The dependence of ρ and β on Δ follows from (5) and (10). The propagation constant $\bar{\beta}$ is given by $\bar{\beta} = \beta_0 - \theta_1$ (or $\Delta = 0$). It is more convenient, however, to express it in terms of the angle α at which the radiation is leaving the slab core.[3] We have

$$\bar{\beta} = n_2 k \cos \alpha \tag{15}$$

and, similarly,

$$\bar{\rho} = n_2 k \sin \alpha. \tag{16}$$

The radiation field that is caused by scattering of light from the even TE guided mode by a sinusoidal variation of the core thickness is thus simply a plane wave.

$$E_y = -\frac{a_1 k^2 (2\omega\mu_o P)^{\frac{1}{2}} (n_1^2 - n_2^2) \cos \kappa_1 d \cos \bar{\sigma} d}{2\left\{\left(\beta_1 d + \dfrac{\beta_1}{\gamma_1}\right)(\bar{\rho}^2 \cos^2 \bar{\sigma} d + \bar{\sigma}^2 \sin \bar{\sigma}^2 d)\right\}^{\frac{1}{2}}} e^{i(\bar{\rho} d - \phi)} e^{-i(\bar{\beta} z + \bar{\rho} x_1)}. \tag{17}$$

The phase terms ϕ is given by (3), the parameter $\bar{\sigma}$ follows from (6) by a replacement of β with $\bar{\beta}$ of (15).

This calculation proves once more that the standing-wave radiation modes result in traveling waves when properly superimposed.

We have so far ignored the finite width of the cladding. We could easily take reflections and refraction at the cladding boundary into account. However, our theory is meant as an order-of-magnitude estimate of crosstalk so that the small reflection losses at the cladding boundary will simply be ignored.

As the radiation reaches the core of the neighboring slab waveguide, reflection and refraction are taking place. It is intuitively clear that the plane wave impinging on the core of the neighboring slab waveguide can be expressed as a superposition of an even and an odd radiation mode of the guide. In fact, it can be shown that the plane wave impinging on the second guide can be expressed by

$$E_y = G\{e^{-i\phi} \mathcal{E}_y^{(e)}(\bar{\rho}) - e^{-i\Psi} \mathcal{E}_y^{(o)}(\bar{\rho})\}, \tag{18}$$

with the even and odd radiation modes (1) and (2) being expressed in a coordinate system centered at the core of the second guide. The coefficient is given by

$$G = -e^{-i\phi} \frac{a_1 k^2 (\pi\beta)^{\frac{1}{2}} (n_1^2 - n_2^2) \cos \kappa_1 d \cos \bar{\sigma} d}{2\left\{\left(\beta_1 d + \dfrac{\beta_1}{\gamma_1}\right)(\bar{\rho}^2 \cos^2 \bar{\sigma} d + \bar{\sigma}^2 \sin^2 \bar{\sigma} d)\right\}^{\frac{1}{2}}} e^{-i\bar{\rho}(R-2d)}. \tag{19}$$

R is the distance between the centers of the two waveguides. We now know how the radiation originating at the first guide excites the radiation modes of the second guide. If we assume that the second guide also has no other imperfections than thickness changes of the core and if we are concerned with the excitation of another even TE mode, only even radiation modes of the second guide can contribute to the crosstalk problem.

The question of how an even radiation mode excites an even guided TE mode has been solved (in principle) in Ref. 2. The excitation of the guided mode is expressed by an excitation coefficient $c(z)$. How-

ever, we are interested only in the value of c at $z = L$. To the first order of perturbation theory we obtain

$$c(L) = ie^{-i2\bar{\rho}(D-d)}e^{-i\rho_{3r}(R-2D)}e^{-\alpha_3(n_3k/\rho_{3r})(R-2D)}e^{-2i\phi}$$
$$\cdot \frac{a_1a_2\bar{\rho}k^4(n_1^2 - n_2^2)^2 \cos\kappa_1 d \cos\kappa_2 d \cos^2\bar{\sigma}d e^{-\alpha_2 L}}{2\left\{\left(\beta_1 d + \frac{\beta_1}{\gamma_1}\right)\left(\beta_2 d + \frac{\beta_2}{\gamma_2}\right)\right\}^{\frac{1}{2}}(\bar{\rho}^2\cos^2\bar{\sigma}d + \bar{\sigma}^2\sin^2\bar{\sigma}d)}$$
$$\cdot \int_0^L e^{-(\alpha_1-\alpha_2)z}\left(\sin\theta_2 z\right)e^{i(\beta_2-\bar{\beta})z}\,dz. \tag{20}$$

Several new parameters have been introduced in (20) (see Fig. 1). R is the separation between the core centers of the two guides. The first exponential factor of (20) accounts for the phase shift of the plane wave inside the claddings of the two guides with cladding thickness $D - d$. The second exponential factor accounts for the phase shift in the medium of index n_3 between the two guides (separation $R - 2D$). The value of $\bar{\rho}$ in the medium between the guides was designated by ρ_{3r} (see Fig. 1). The subscript r indicates the real part of ρ_3. The imaginary part of ρ_3 attenuates the plane wave as it traverses the space between the two guides. We have assumed that the core and cladding of both guides are lossless but allow for the possibility that the medium separating the two guides may be lossy. The loss suffered by the plane wave in the medium between the guides is given by the third exponential function in (20). The real part of the refractive index of the medium between the guides is n_3, the amplitude loss coefficient of the wave is α_3. The ratio n_3k/ρ_{3r} adjusts for the slant angle of the wave and $R - 2D$ is the separation of the two guides (cladding-to-cladding distance). The subscripts 1 and 2 refer to quantities belonging to guide 1 and 2. Finally, we have added the amplitude loss coefficients α_1 and α_2 to account for the losses suffered by each guided mode. These losses can be thought of as the sum of heat losses in each guide plus radiation losses caused by the variation of the core thickness. The factors a_1 and a_2 indicate the amplitudes of the sinusoidal thickness variation of the cores of each guide. However, it is assumed that the perfect geometry and the refractive indices of both guides are identieal. The mechanical frequencies θ_1 and θ_2 are not independent of each other. Only when the propagation constant β_2 of the guided mode of the second guide and the mechanical frequency θ_2 of the thickness variation of its core are related to each other by the equation

$$\theta_2 = \beta_2 - \bar{\beta} \tag{21}$$

can any amount of power be coupled from one guide to the other.

From (10) (with $\beta = \bar{\beta}$ and $\Delta = 0$) and (21) we obtain the condition

$$\theta_2 = \theta_1 - (\beta_1 - \beta_2). \tag{22}$$

The range of θ_1 values contributing to radiation is given by[3]

$$\beta_1 - n_2 k < \theta_1 < \beta_1 + n_2 k. \tag{23}$$

In the more general case of a Fourier spectrum of mechanical frequencies of the core thickness variation, we see that each Fourier component of guide 2 combines with a certain Fourier component of guide 1 to provide crosstalk.

III. GENERAL CORE THICKNESS VARIATION

Having determined the amplitude coefficient of the guided mode in guide 2 that is excited by purely sinusoidal variations of the thickness of the fiber cores we can immediately generalize to the case of arbitrary core thickness variation. We expand the core thickness of each guide in a Fourier series:

$$d_1 = d + \left(\frac{2}{L}\right)^{\frac{1}{2}} \sum_{\nu=1}^{\infty} a_{1\nu} \sin \theta_\nu z, \tag{24}$$

$$d_2 = d + \left(\frac{2}{L}\right)^{\frac{1}{2}} \sum_{\mu=1}^{\infty} a_{2\mu} \sin \theta_\mu z, \tag{25}$$

with $\theta_\nu = \nu\pi/L$. We choose the Fourier sine series because we already know the result for a single sinusoidal thickness variation. The expansions (24) and (25) involve a complete set of orthogonal functions. By replacing

$$a_1 \rightarrow \left(\frac{2}{L}\right)^{\frac{1}{2}} a_{1\nu} \tag{26}$$

and

$$a_2 \rightarrow \sqrt{\frac{2}{L}}\, a_{2\mu}\ , \tag{27}$$

and summing over ν we convert (20) to the general case. Only one summation is required since each Fourier component of guide 1 combines with only one definite Fourier component of guide 2. However, before writing down the resulting formula we will make two more changes. *First*, we can convert the sum to an integral with the help of the relation

$$\sum_{\nu} \rightarrow \frac{L}{\pi} \int d\theta. \tag{28}$$

Second, we introduce the Fourier coefficient $\phi(\theta)$ that was used to express the radiation losses in Ref. 3. By definition we have

$$\phi_j(\theta) = \frac{1}{L}\int_0^L (d_j - d)e^{-i\theta z}\,dz \qquad j = 1, 2. \tag{29}$$

The relation between $\phi_j(\theta)$ and $a_{j\nu}$ is given by the following equation

$$\phi_j^{(i)}(\theta_\nu) = \operatorname{Im}\,[\phi_j(\theta_\nu)] = -\frac{1}{(2L)^{\frac{1}{2}}}\,a_{j\nu} \qquad j = 1, 2. \tag{30}$$

The notation Im indicates the imaginary part. Then we form the ratio of power at the end of guide 2, $P_2(L)$, to the power at the end of guide 1, $P_1(L) = e^{-2\alpha_1 L}P$,

$$\frac{P_2(L)}{P_1(L)} = \frac{|\,c\,|^2\,P}{P_1(L)} = e^{2\alpha_1 L}\,|\,c\,|^2. \tag{31}$$

With the help of (21) the integral in (20) can easily be performed and we finally obtain

$$\frac{P_2(L)}{P_1(L)} = \frac{L^2(n_1^2 - n_2^2)^4k^8\cos^2\kappa_1 d\cos^2\kappa_2 d e^{2(\alpha_1-\alpha_2)L}}{\pi^2\left(\beta_1 d + \frac{\beta_1}{\gamma_1}\right)\left(\beta_2 d + \frac{\beta_2}{\gamma_2}\right)}\left(\frac{1 - e^{-(\alpha_1-\alpha_2)L}}{\alpha_1 - \alpha_2}\right)^2$$
$$\cdot\left|\int_{\beta_1 - n_2 k}^{\beta_1 + n_2 k}\frac{\phi_1^{(i)}(\theta_1)\phi_2^{(i)}(\theta_2)\bar{\rho}\cos^2\bar{\sigma}d}{\bar{\rho}^2\cos^2\bar{\sigma}d + \bar{\sigma}^2\sin^2\bar{\sigma}d}\right.$$
$$\left.\cdot e^{-2i\phi}e^{-i[2\bar{\rho}(D-d)+\rho_{2r}(R-2D)]}e^{-\alpha_2(n_2k/\rho_{2r})(R-2D)}\,d\theta_1\right|^2. \tag{32}$$

Equation (32) is the expression for the crosstalk between two identical slab waveguides. The coupling mechanism is the thickness variation of the two waveguide cores. The even guided TE modes in either waveguide need not be the same.

In its general form the crosstalk ratio is not very useful since the spectral functions ϕ_1 and ϕ_2 are usually not known. However, it is possible to provide estimates of the crosstalk by using the radiation loss of the guided mode that is caused by the same mechanism that provides the crosstalk. The scattering loss was given in Ref. 3 $(i = 1, 2)$.

$$2\alpha_{ri} = \frac{\Delta P_i}{PL} = \frac{L(n_1^2 - n_2^2)^2k^4\cos^2\kappa_i d}{\pi\left(\beta_i d + \frac{\beta_i}{\gamma_i}\right)}$$
$$\cdot\int_{-n_2k}^{n_2k}\frac{|\,\phi_i(\theta_i)\,|^2\,\rho\cos^2\sigma d}{\rho^2\cos^2\sigma d + \sigma^2\sin^2\sigma d}\,d\beta. \tag{33}$$

[There is an error in equation (14a), Ref. 3. The left-hand side should read $\Delta Pd/(PL)$.] Equation (33) differs from equation (14) of Ref. 3 since we have here assumed that the core thickness varies while in Ref. 3 it was assumed that only one side of the core cladding interface is distorted.

For simplicity we consider now the case that the two modes of the coupled guides are identical and that only one sine wave component of the Fourier spectrum exists. We thus have ($\phi_1 = \phi_2$):

$$|\phi_1(\theta)|^2 = |\phi_2(\theta)|^2 = \frac{\pi}{2L} a^2 \delta(\theta - \theta') \tag{34a}$$

and

$$[\phi_1^{(i)}(\theta)]^2 = \phi_1^{(i)}(\theta)\phi_2^{(i)}(\theta) = \frac{\pi}{4L} a^2 \delta(\theta - \theta'), \tag{34b}$$

and obtain from (32) and (33) with $\alpha_1 = \alpha_2$

$$\frac{P_2(L)}{P_1(L)} = (1/4)(2\alpha_r L)^2 e^{-2\alpha_3(n_3k/\rho_{3r})(R-2D)}. \tag{35}$$

The index r is meant to indicate that α_r is the radiation loss. In this special case the crosstalk is simply proportional to the square of the mode power loss coefficient $2\alpha_r L$ times the power loss that the plane wave suffers in going from guide 1 to guide 2 through the lossy medium between the guides.

IV. RANDOM VARIATIONS OF THE CORE THICKNESS

The average radiation loss is simply obtained by replacing $|\phi_i|^2$ with the ensemble average $\langle|\phi_i|^2\rangle$ in (33). Obtaining the ensemble average of (32) is a little more complicated. To simplify the discussion we write the integral expression occurring in (32) in the following form

$$I = \left| \int_{\beta_1 - n_3k}^{\beta_1 + n_3k} \phi_1^{(i)}(\theta_1)\phi_2^{(i)}(\theta_2)F(\theta)\, d\Theta \right|^2$$
$$= \left| \frac{\pi}{L} \sum_{\nu=1}^{\infty} \phi_1^{(i)}(\theta_\nu)\phi_2^{(i)}(\theta_\mu)F(\theta_\nu) \right|^2, \tag{36}$$

with

$$\theta_\mu = \theta_\nu - (\beta_1 - \beta_2). \tag{37}$$

With the help of (30) we can also write for the ensemble average of I

$$\langle I \rangle = \frac{\pi^2}{4L^4} \sum_{\nu\nu'} \langle a_{1\nu} a_{1\nu'}^* a_{2\mu} a_{2\mu'}^* \rangle F(\theta_\nu) F^*(\theta_{\nu'}). \tag{38}$$

The asterisk indicates complex conjugation. It appears reasonable to assume that there is no correlation between guide 1 and 2. We thus assume that we can write

$$\langle a_{1\nu} a_{1\nu'}^* a_{2\mu} a_{2\mu'}^* \rangle = \langle a_{1\nu} a_{1\nu'}^* \rangle \langle a_{2\mu} a_{2\mu'}^* \rangle. \tag{39}$$

It is shown in Ref. 4 that for a stationary random process with $L \to \infty$ we have

$$\langle a_{1\nu} a_{1\nu'}^* \rangle = \langle \, |a_{1\nu}|^2 \rangle \delta_{\nu\nu'} \, . \tag{40}$$

[The proof presented in Ref. 4 for the usual complex Fourier series can easily be extended to the Fourier series (24). The conclusion (40) remains correct.]

We thus have

$$\langle I \rangle = \frac{\pi^2}{4L^4} \sum_{\nu} \langle | \, a_{1\nu} \, |^2 \rangle \langle | \, a_{2\mu} \, |^2 \rangle \, | \, F(\theta_\nu) \, |^2. \tag{41}$$

Returning to the integral notation we have

$$\langle I \rangle = \frac{\pi}{L} \int_{\beta_1 - n_2 k}^{\beta_1 + n_2 k} \langle | \, \phi_1^{(i)}(\theta_1) \, |^2 \rangle \langle | \, \phi_2^{(i)}(\theta_2) \, |^2 \rangle \, | \, F(\theta_1) \, |^2 \, d\theta_1 \, . \tag{42}$$

For stationary random processes we can assume $\langle |\phi^{(i)}(\theta)|^2 \rangle = \frac{1}{2} \langle |\phi(\theta)|^2 \rangle$. We thus obtain the ensemble average of the crosstalk in the form:

$$\frac{\langle P_2(L)/P_1(L) \rangle}{\langle 2\alpha_{r1} L \rangle \langle 2\alpha_{r2} L \rangle}$$
$$= \frac{\pi}{4n_2 k L} \frac{e^{2(\alpha_1 - \alpha_2)L} \left(\dfrac{1 - e^{-(\alpha_1 - \alpha_2)L}}{\alpha_1 - \alpha_2} \right)^2 \dfrac{1}{L^2} \displaystyle\int_0^\pi \frac{\langle | \, \phi_1(\theta_1) \, |^2 \rangle \langle | \, \phi_2(\theta_2) \, |^2 \rangle \bar{\rho}^3 n_2 k \cos^4 \bar{\sigma} d}{(\bar{\rho}^2 \cos^2 \sigma d + \bar{\sigma}^2 \sin^2 \bar{\sigma} d)^2} e^{-2\alpha_2 (n_2 k/\rho_{2\,r})(R - 2d)} \, d\alpha}{\displaystyle\int_0^\pi \frac{\langle | \, \phi_1(\theta) \, |^2 \rangle \rho^2 \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} \, d\alpha \int_0^\pi \frac{\langle | \, \phi_2(\theta) \, |^2 \rangle \rho^2 \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} \, d\alpha} . \tag{43}$$

We used (15), (16), and (10) which, with $\Delta = 0$, assumes the form

$$\theta_1 = \beta_1 - \bar{\beta} = \beta_1 - n_2 k \cos \alpha \tag{44}$$

to obtain

$$d\theta_1 = n_2 k \sin \alpha d\alpha = \bar{p} d\alpha$$

for the integral in the numerator of (43). The angle α indicates the direction in which the plane wave radiation from a given Fourier

component escapes from the waveguide core. (The angle α must not be confused with the loss coefficients α_1 or α_r.) In the integrals in the denominator we have likewise used the transformation

$$\beta = n_2 k \cos \alpha, \tag{45}$$

so that we obtain from (5)

$$\rho = (n_2^2 k^2 - \beta^2)^{\frac{1}{2}} = n_2 k \sin \alpha \tag{46}$$

and

$$d\beta = -\rho d\alpha. \tag{47}$$

For a discussion of (43) it is much more convenient to consider the case that mode 1 of guide 1 and mode 2 of guide 2 are identical. The conclusions for the more general case are very similar. We can furthermore assume that the power spectra of the two guides are identical since they are supposed to be statistically similar,

$$\langle |\phi_1|^2 \rangle = \langle |\phi_2|^2 \rangle = \langle |\phi|^2 \rangle. \tag{48}$$

We thus obtain the much simplier formula (For simplicity we use $\alpha_3 = 0$)

$$\frac{\langle P_2(L)/P_1(L) \rangle}{\langle 2\alpha_r L \rangle^2} = \frac{\pi}{4n_2 kL} \frac{\displaystyle\int_0^\pi \frac{\langle |\phi(\theta)|^2 \rangle^2 n_2 k \rho^3 \cos^4 \sigma d}{(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)^2} \, d\alpha}{\left\{ \displaystyle\int_0^\pi \frac{\langle |\phi(\theta)|^2 \rangle \rho^2 \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} \, d\alpha \right\}^2}. \tag{49}$$

V. DISCUSSION OF THE SCATTERING CROSSTALK

The crosstalk for identical modes, equation (49), can be expressed approximately by very simple formulas. Let us assume that the Fourier spectrum of the core thickness variation is so narrow that only a narrow beam of radiation leaves guide 1. The radiation is centered around an angle α and fills a range $\Delta\alpha$. Provided this range of angles is sufficiently narrow and if we also assume that the Fourier spectrum is constant in the region contributing radiation and zero outside of this region, we can write (49) in the following simple form.

$$\left\langle \frac{P_2(L)}{P_1(L)} \right\rangle = \frac{\pi}{4n_2 kL} \frac{1}{(\Delta\alpha) \sin \alpha} \langle 2\alpha_r L \rangle^2 e^{-2\alpha_3 (R-2D)/\sin \alpha}. \tag{50}$$

(The angle α and the loss coefficients α_r and α_3 must not be confused.) We have reinserted the loss in the medium between the two guides in (50). We thus have the interesting result that the crosstalk

depends on the square of the loss coefficient of the guided mode, on the loss in the medium between the waveguides, and on the width of the radiation lobe. The crosstalk decreases with increasing lobe width. In fact, by comparing (50) with (35) we see that we can let $\Delta\alpha$ become as small as

$$\Delta\alpha = \frac{\pi}{n_2 kL \sin \alpha}, \tag{51}$$

because in this limit (50) applies to a single plane wave coupling the two guides together.

In the other extreme it can be shown that (49) can be approximated quite well by the expression

$$\left\langle \frac{P_2(L)}{P_1(L)} \right\rangle = \frac{3}{4n_2 kL} \langle 2\alpha_r L \rangle^2, \tag{52}$$

if we assume that $\langle |\phi|^2 \rangle$ = const, which corresponds to the case of random scattering of radiation in all directions. We assumed $\alpha_3 = 0$ in (52). The error between the approximation (52) and the equation (49) is always less than a factor of two. This is not a bad approximation for an order-of-magnitude estimate. It is interesting to note that we obtain (52) formally from (50) by taking

$$\Delta\alpha \sin \alpha = \frac{\pi}{3}. \tag{53}$$

Since it is certainly reasonable to assign $\Delta\alpha = \pi$ to the case of isotropic radiation, we see that the crude approximation (50), which holds precisely only when $\Delta\alpha$ is very small, can actually be used to estimate the scattering crosstalk for all cases of interest.

It is furthermore important to note that the model of scattering used to derive our equations is very likely to be of no importance to the results. Since we could express the crosstalk in terms of the radiation loss of the guided mode and some features of the radiation spectrum we can be confident that our equation (50) holds at least to order of magnitude for any type of waveguide imperfection that causes radiation.

If we ask ourselves what relation the slab waveguide theory may have to the case of round optical fibers we can only draw some very general conclusions. The plane wave radiation produced by the slab waveguide does not diminish with distance. It is thus not surprising that the crosstalk formula (50) is independent of distance except for the obvious distance dependence of the loss term containing α_3. The

radiation of a round optical fiber has the form of cylindrical waves so that we must expect that, in addition to the features expressed by (50), there will be an additional factor d/R multiplying the crosstalk expression. It is hard to say whether this adjustment will yield an expression that describes the crosstalk for round fibers to a fair approximation. However, it appears safe to assume that the crosstalk predicted by the slab waveguide theory is actually larger than the crosstalk for round optical fibers. We can thus consider the results of this paper as an upper bound of the crosstalk of round optical fibers. Any design features that are predicted on the basis of this theory are likely to be conservative if applied to the round fiber case.

We conclude our discussion with a numerical example. Let us assume that the length of the guides is $L = 1$ km. Taking $d = 1$ μm, we have $L/d = 10^9$. Next, we assume that $n_2kd = 10$ and that the radiation loss of the waveguides is 4 dB/km corresponding to $2\alpha_1 L = 1$. In the absence of loss in the medium between the two guides ($\alpha_3 = 0$) we thus obtain from (52)

$$\left\langle \frac{P_2(L)}{P_1(L)} \right\rangle = 3/4 \times 10^{-10}. \tag{54}$$

The crosstalk is thus insignificant even without trying to isolate the guides from each other by providing loss in the medium between the guides. This example is actually quite representative. The assumed mode loss is typical for Rayleigh scattering losses in good solid materials. Since Rayleigh scattering is the theoretical limit of scattering loss, one should consider the crosstalk caused by this mechanism. Our conclusion is thus that Rayleigh scattering does not cause appreciable crosstalk between two optical fibers. The situation worsens somewhat if the radiation is bunched. However, we can stand a much smaller value of $\Delta\alpha$ than $\Delta\alpha = \pi$ before crosstalk becomes a problem. It appears that the only real danger is the possibility of a pure sinusoidal thickness variation of the fiber core such as may be caused by a systematic flaw in the fiber pulling process. Such an imperfection can cause serious crosstalk even if the radiation loss attributable to this mechanism is quite small. We see from (35) that a purely sinusoidal distortion of the waveguide core causing only $2\alpha_1 L = 10^{-2}$ can cause a crosstalk ratio of 2.5×10^{-5}. The theory of Reference 2 predicts that a loss of $2\alpha_1 L = 10^{-2}$ is caused by a sinusoidal core thickness variation with an amplitude as small as $\approx 10^{-5}$ μm.

VI. CONCLUSIONS

Using the model of two slab waveguides with core thickness variations we have computed the crosstalk that is caused by light scattering in one guide and reconversion of this scattered light in the other guide. Since the results of the theory can be expressed in terms of the radiation loss that is caused by the same scattering mechanism, it is assumed that the theory has general validity. It is also concluded that the crosstalk of round optical fibers is smaller than that of two slab waveguides so that the theory can be considered as an upper bound on the crosstalk between fibers.

Scattering crosstalk can be substantial if both waveguides have a sinusoidal imperfection of equal mechanical frequency that persists throughout the entire length of the guides. If such an imperfection should exist, it would be necessary to isolate the waveguides from each other by providing a surrounding medium with loss. However, it appears unlikely that such a systematic sinusoidal deformation should exist since it would require that a definite mechanical period with a frequency inside of the range given by (23) should have been generated by the manufacturing process.

Any other imperfections that are of a statistical nature are much less serious even though they might lead to radiation patterns with narrow radiation lobes. It appears likely that scattering crosstalk will not present a problem particularly since the radiation losses of the waveguides must be kept low in order for the guides to be useful. Unless a definite sinusoidal imperfection does exist it seems unnecessary to provide loss in the medium separating the two waveguides in order to reduce the scattering crosstalk.

REFERENCES

1. Marcuse, D., "The Coupling of Degenerate Modes in Two Parallel Dielectric Waveguides," B.S.T.J., this issue, pp. 1791–1816.
2. Marcuse, D., "Mode Conversion Caused by Surface Imperfections of a Dielectric Slab Waveguide," B.S.T.J., *48,* No. 10, part 3 (December 1969), pp. 3187–3215.
3. Marcuse, D., "Radiation Losses of Dielectric Waveguides in Terms of the Power Spectrum of the Wall Distortion Function," B.S.T.J., *48,* No. 10, part 3 (December 1969), pp. 3233–3242.
4. Rowe, H. E., *Signals and Noise in Communications Systems,* New York: D. Van Nostrand Company, 1965.